

STABILITY OF EXTREMAL KÄHLER MANIFOLDS

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Dedicated to Professor Shoshichi Kobayashi on his seventieth birthday

1. INTRODUCTION

In Donaldson's study [10] of asymptotic stability for polarized algebraic manifolds (M, L) , *critical metrics* originally defined by Zhang [39] (see also [22]) are referred to as balanced metrics and play a central role when the polarized algebraic manifolds admit Kähler metrics of constant scalar curvature. Let $T \cong (\mathbb{C}^*)^k$ be an algebraic torus in the identity component $\text{Aut}^0(M)$ of the group of holomorphic automorphisms of M . In this paper, we define the concept of *critical metrics relative to T* , and as an application, choosing a suitable T , we shall show that a result in [26] on the asymptotic approximation of critical metrics (see [10], [39]) can be generalized to the case where (M, L) admits an extremal Kähler metric in the polarization class. Then in our forthcoming paper [27], we shall show that a slight modification of the concept of stability (see Theorem A below) allows us to obtain the asymptotic stability of extremal Kähler manifolds even when the obstruction as in [26] does not vanish. In particular, by an argument similar to [10], an extremal Kähler metric in a fixed integral Kähler class on a projective algebraic manifold M will be shown to be unique[†] up to the action of the group $\text{Aut}^0(M)$.

2. STATEMENT OF RESULTS

Throughout this paper, we fix once for all an ample holomorphic line bundle L on a connected projective algebraic manifold M . Let H be the maximal connected linear algebraic subgroup of $\text{Aut}^0(M)$, so that $\text{Aut}^0(M)/H$ is an abelian variety. The corresponding Lie subalgebra of $H^0(M, \mathcal{O}(T^{1,0}M))$ will be denoted by \mathfrak{h} . For the complete linear system $|L^m|$, $m \gg 1$, we consider the Kodaira embedding

$$\Phi_m = \Phi_{|L^m|} : M \hookrightarrow \mathbb{P}^*(V_m), \quad m \gg 1,$$

where $\mathbb{P}^*(V_m)$ denotes the set of all hyperplanes through the origin in $V_m := H^0(M, \mathcal{O}(L^m))$. Put $N_m := \dim V_m - 1$. Let n and d be respectively the dimension of M and the degree of the image $M_m := \Phi_m(M)$ in the projective space $\mathbb{P}^*(V_m)$. Put $W_m = \{\text{Sym}^d(V_m)\}^{\otimes n+1}$. Then to the image M_m of M , we can associate a nonzero element \hat{M}_m in W_m^* such that the corresponding element $[\hat{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point associated to the irreducible

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[†]For this uniqueness, we choose $Z^{\mathbb{C}}$ (cf. Section 2) as the algebraic torus T .

reduced algebraic cycle M_m on $\mathbb{P}^*(V_m)$. Replacing L by some positive integral multiple of L if necessary, we fix an H -linearization of L , i.e., a lift to L of the H -action on M such that H acts on L as bundle isomorphisms covering the H -action on M . For an algebraic torus T in H , this naturally induces a T -action on V_m for each m . Now for each character $\chi \in \text{Hom}(T, \mathbb{C}^*)$, we set

$$V(\chi) := \{s \in V_m ; t \cdot s = \chi(t)s \text{ for all } t \in T\}.$$

Then we have mutually distinct characters $\chi_1, \chi_2, \dots, \chi_{\nu_m} \in \text{Hom}(T, \mathbb{C}^*)$ such that the vector space $V_m = H^0(M, \mathcal{O}(L^m))$ is uniquely written as a direct sum

$$(2.1) \quad V_m = \bigoplus_{k=1}^{\nu_m} V(\chi_k).$$

Put $G_m := \prod_{k=1}^{\nu_m} \text{SL}(V(\chi_k))$, and the associated Lie subalgebra of $\text{sl}(V_m)$ will be denoted by \mathfrak{g}_m . More precisely, G_m and \mathfrak{g}_m possibly depend on the choice of the algebraic torus T , and if necessary, we denote these by $G_m(T)$ and $\mathfrak{g}_m(T)$, respectively. The T -action on V_m is, more precisely, a right action, while we regard the G_m -action on V_m as a left action. Since T is Abelian, this T -action on V_m can be regarded also as a left action.

The group G_m acts diagonally on V_m in such a way that, for each k , the k -th factor $\text{SL}(V(\chi_k))$ of G_m acts just on the k -th factor $V(\chi_k)$ of V_m . This induces a natural G_m -action on W_m and also on W_m^* .

Definition 2.2. (a) The subvariety M_m of $\mathbb{P}^*(V_m)$ is said to be *stable relative to T* or *semistable relative to T* , according as the orbit $G_m \cdot \hat{M}_m$ is closed in W_m^* or the closure of $G_m \cdot \hat{M}_m$ in W_m^* does not contain the origin of W_m^* .

(b) Let \mathfrak{t}_c denote the Lie subalgebra of the maximal compact subgroup T_c of T , and as a real Lie subalgebra of the complex Lie algebra \mathfrak{t} , we define $\mathfrak{t}_{\mathbb{R}} := \sqrt{-1} \mathfrak{t}_c$.

Take a Hermitian metric for V_m such that $V(\chi_k) \perp V(\chi_\ell)$ if $k \neq \ell$. Put $N_m := \dim V_m - 1$ and $n_k := \dim V(\chi_k)$. We then set

$$l(k, i) := (i - 1) + \sum_{j=1}^{k-1} n_j, \quad i = 1, 2, \dots, n_k; \quad k = 1, 2, \dots, \nu_m,$$

where the right-hand side denotes $i - 1$ in the special case $k = 1$. Let $\| \cdot \|$ denote the Hermitian norm for V_m induced by the Hermitian metric. Take a \mathbb{C} -basis $\{s_0, s_1, \dots, s_{N_m}\}$ for V_m .

Definition 2.3. We say that $\{s_0, s_1, \dots, s_{N_m}\}$ is an *admissible normal basis* for V_m if there exist positive real constants b_k , $k = 1, 2, \dots, \nu_m$, and a \mathbb{C} -basis $\{s_{k,i}; i = 1, 2, \dots, n_k\}$ for $V(\chi_k)$, with $\sum_{k=1}^{\nu_m} n_k b_k = N_m + 1$, such that

- (1) $s_{l(k,i)} = s_{k,i}$, $i = 1, 2, \dots, n_k; \quad k = 1, 2, \dots, \nu_m;$
- (2) $s_l \perp s_{l'} \text{ if } l \neq l';$

$$(3) \|s_{k,i}\|^2 = b_k, \quad i = 1, 2, \dots, n_k; \quad k = 1, 2, \dots, \nu_m.$$

Then the real vector $b := (b_1, b_2, \dots, b_{\nu_m})$ is called the *index* of the admissible normal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for V_m .

We now specify a Hermitian metric on V_m . For the maximal compact subgroup T_c of T above, let \mathcal{S} be the set ($\neq \emptyset$) of all T_c -invariant Kähler forms in the class $c_1(L)_{\mathbb{R}}$. Let $\omega \in \mathcal{S}$, and choose a Hermitian metric h for L such that $\omega = c_1(L; h)$. Define a Hermitian metric on V_m by

$$(2.4) \quad (s, s')_{L^2} := \int_M (s, s')_{h^m} \omega^n, \quad s, s' \in V_m,$$

where $(s, s')_{h^m}$ denotes the function on M obtained as the pointwise inner product of s, s' by the Hermitian metric h^m on L^m . Now, let us consider the situation that V_m has the Hermitian metric (2.4). Then

$$V(\chi_k) \perp V(\chi_\ell), \quad k \neq \ell,$$

and define a maximal compact subgroup $(G_m)_c$ of G_m by $(G_m)_c := \prod_{k=1}^{\nu_m} \mathrm{SU}(V(\chi_k))$. Again by this Hermitian metric $(\cdot, \cdot)_{L^2}$, let $\{s_0, s_1, \dots, s_{N_m}\}$ an admissible normal basis for V_m of a given index b . Put

$$(2.5) \quad E_{\omega, b} := \sum_{i=0}^{N_m} |s_i|_{h^m}^2,$$

where $|s|_{h^m} := (s, s)_{h^m}$ for all $s \in V_m$. Then $E_{\omega, b}$ depends only on ω and b . Namely, once ω and b are fixed, $E_{\omega, b}$ is independent of the choice of an admissible normal basis for $V(\chi_k)$ of index b . Fix a positive integer m such that L^m is very ample.

Definition 2.6. An element ω in \mathcal{S} is called a *critical metric relative to T* , if there exists an admissible normal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for V_m such that the associated function $E_{\omega, b}$ on M is constant for the index b of the admissible normal basis. This generalizes a *critical metric* of Zhang [39] (see also [5]) who treated the case $T = \{1\}$. If ω is a critical metric relative to T , then by integrating the equality (2.5) over M , we see that the constant $E_{\omega, b}$ is $(N_m + 1)/c_1(L)^n[M]$.

For the centralizer $Z_H(T)$ of T in H , let $Z_H(T)^0$ be its identity component. For m as above, the following generalization of a result in [39] is crucial to our study of stability:

Theorem A. *The subvariety M_m of $\mathbb{P}(V_m)$ is stable relative to T if and only if there exists a critical metric $\omega \in \mathcal{S}$ relative to T . Moreover, for a fixed index b , a critical metric ω in \mathcal{S} relative to T with constant $E_{\omega, b}$ is unique up to the action of $Z_H(T)^0$.*

We now fix a maximal compact connected subgroup K of H . The corresponding Lie subalgebra of \mathfrak{h} is denoted by \mathfrak{k} . Let \mathcal{S}_K denote the set of all Kähler forms ω in the class $c_1(L)_{\mathbb{R}}$ such that the identity component of the group of the isometries of (M, ω) coincides

with K . Then $\mathcal{S}_K \neq \emptyset$, and an extremal Kähler metric, if any, in the class $c_1(L)_\mathbb{R}$ is always in H -orbits of elements of \mathcal{S}_K . For each $\omega \in \mathcal{S}_K$, we write

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha, \beta} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$$

in terms of a system (z^1, \dots, z^n) of holomorphic local coordinates on M . let \mathcal{K}_ω be the space of all real-valued smooth functions u on M such that $\int_M u \omega^n = 0$ and that

$$\text{grad}_\omega^\mathbb{C} u := \frac{1}{\sqrt{-1}} \sum_{\alpha, \beta} g^{\bar{\beta}\alpha} \frac{\partial u}{\partial z^\beta} \frac{\partial}{\partial z^\alpha}$$

is a holomorphic vector field on M . Then \mathcal{K}_ω forms a real Lie subalgebra of \mathfrak{h} by the Poisson bracket for (M, ω) . We then have the Lie algebra isomorphism

$$\mathcal{K}_\omega \cong \mathfrak{k}, \quad u \leftrightarrow \text{grad}_\omega^\mathbb{C} u.$$

For the space $C^\infty(M)_\mathbb{R}$ of real-valued smooth functions on M , we consider the inner product defined by $(u_1, u_2)_\omega := \int_M u_1 u_2 \omega^n$ for $u_1, u_2 \in C^\infty(M)_\mathbb{R}$. Let $\text{pr} : C^\infty(M)_\mathbb{R} \rightarrow \mathcal{K}_\omega$ be the orthogonal projection. Let \mathfrak{z} be the center of \mathfrak{k} . Then the vector field

$$\mathcal{V} := \text{grad}_\omega^\mathbb{C} \text{pr}(\sigma_\omega) \in \mathfrak{z}$$

is called the *extremal Kähler vector field* of (M, ω) , where σ_ω denotes the scalar curvature of ω . Then \mathcal{V} is independent of the choice of ω in \mathcal{S} , and satisfies $\exp(2\pi\gamma\mathcal{V}) = 1$ for some positive integer γ (cf. [13], [32]). Next, since we have an H -linearization of L , there exists a natural inclusion $H \subset \text{GL}(V_m)$. By passing to the Lie algebras, we obtain

$$\mathfrak{h} \subset \mathfrak{gl}(V_m).$$

Take a Hermitian metric h for L such that the corresponding first Chern form $c_1(L; h)$ is ω . As in [23], (1.4.1), the infinitesimal \mathfrak{h} -action on L induces an infinitesimal \mathfrak{h} -action on the complexification $\mathcal{H}_m^\mathbb{C}$ of the space of all Hermitian metrics \mathcal{H}_m on the line bundle L^m . The Futaki-Morita character $F : \mathfrak{h} \rightarrow \mathbb{C}$ is given by

$$F(\mathcal{Y}) := \frac{\sqrt{-1}}{2\pi} \int_M h^{-1}(\mathcal{Y}h) \omega^n,$$

which is independent of the choice of h (see for instance [15]). For the identity component Z of the center of K , we consider its complexification $Z^\mathbb{C}$ in H . Then the corresponding Lie algebra is just the complexification $\mathfrak{z}^\mathbb{C}$ of \mathfrak{z} above. We now consider the set Δ of all algebraic tori in $Z^\mathbb{C}$. Let $T \in \Delta$. Put

$$q := 1/m.$$

For $\omega = c_1(L; h) \in \mathcal{S}_K$, we consider the Hermitian metric (2.4) for V_m . We then choose an admissible normal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for V_m of index $(1, 1, \dots, 1)$. By the asymptotic

expansion of Tian-Zelditch (cf. [33], [38]; see also [4]) for $m \gg 1$, there exist real-valued smooth functions $a_k(\omega)$, $k = 1, 2, \dots$, on M such that

$$(2.7) \quad \frac{n!}{m^n} \sum_{j=0}^{N_m} |s_j|_{h^m}^2 = 1 + a_1(\omega)q + a_2(\omega)q^2 + \dots$$

Then $a_1(\omega) = \sigma_\omega/2$ by a result of Lu [20]. Let $\mathcal{Y} \in \mathfrak{t}_\mathbb{R}$, and put $g := \exp^\mathbb{C} \mathcal{Y} \in T$, where the element $\exp(\mathcal{Y}/2)$ in T is written as $\exp^\mathbb{C} \mathcal{Y}$ by abuse of terminology. Recall that the T -action on V_m is a right action, though it can be viewed also as a left action. Put $h_g := h \cdot g$ for simplicity. Using the notation in Definition 2.3, we write $s_{k,i} = s_{l(k,i)}$, $k = 1, 2, \dots, \nu_m$; $i = 1, 2, \dots, n_k$. Then for a fixed k , $\int_M |s_{k,i}|_{h_g^m}^2 g^* \omega^n = |\chi_k(\exp^\mathbb{C} \mathcal{Y})|^{-2}$ is independent of the choice of i . Put

$$Z(q, \omega; \mathcal{Y}) := \frac{n!}{m^n} \sum_{j=0}^{N_m} |s_j|_{h_g^m}^2 = g^* \left\{ \frac{n!}{m^n} \sum_{k=1}^{\nu_m} |\chi_k(\exp^\mathbb{C} \mathcal{Y})|^{-2} \sum_{i=1}^{n_k} |s_{k,i}|_{h^m}^2 \right\}, \quad \mathcal{Y} \in \mathfrak{t}_\mathbb{R}.$$

For extremal Kähler manifolds, the following generalization of [26] allows us to approximate arbitrarily some critical metrics relative to T :

Theorem B. *Let $\omega_0 = c_1(L; h_0)$ be an extremal Kähler metric in the class $c_1(L)_\mathbb{R}$ with extremal Kähler vector field \mathcal{V} . Then for some $T \in \Delta$, there exist a sequence of vector fields $\mathcal{Y}_k \in \mathfrak{t}_\mathbb{R}$, a formal power series C_q in q with real coefficients (cf. Section 6), and smooth real-valued functions φ_k , $k = 1, 2, \dots$, on M such that*

$$(2.8) \quad Z(q, \omega(\ell); \mathcal{Y}(\ell)) = C_q + o(q^{\ell+2}),$$

where $\mathcal{Y}(\ell) := (\sqrt{-1} \mathcal{V}/2) q^2 + \sum_{k=1}^\ell q^{k+2} \mathcal{Y}_k$, $h(\ell) := h_0 \exp(-\sum_{k=1}^\ell q^k \varphi_k)$, and $\omega(\ell) := c_1(L; h(\ell))$.

The equality (2.8) above means that there exists a positive real constant A_ℓ independent of q such that $\|Z(q, \omega(\ell); \mathcal{Y}(\ell)) - C_q\|_{C^0(M)} \leq A_\ell q^{\ell+2}$ for all q with $0 \leq q \leq 1$. By [38], for every nonnegative integer j , a choice of a larger constant $A = A_{j,\ell} > 0$ keeps Theorem B still valid even if the $C^0(M)$ -norm is replaced by the $C^j(M)$ -norm.

3. A STABILITY CRITERION

In this section, some stability criterion will be given as a preliminary. In a forthcoming paper [27], we actually use a stronger version of Theorem 3.2 which guarantees the stability only by checking the closedness of orbits through a point for special one-parameter subgroups “perpendicular” to the isotropy subgroup. Now, for a connected reductive algebraic group G , defined over \mathbb{C} , we consider a representation of G on an N -dimensional complex vector space W . We fix a maximal compact subgroup G_c of G . Moreover, let \mathbb{C}^* be a one-dimensional algebraic torus with the maximal compact subgroup S^1 .

Definition 3.1. (a) An algebraic group homomorphism $\lambda : \mathbb{C}^* \rightarrow G$ is said to be a *special one-parameter subgroup* of G , if the image $\lambda(S^1)$ is contained in G_c .

(b) A point $w \neq 0$ in W is said to be *stable*, if the orbit $G \cdot w$ is closed in W .

Later, we apply the following stability criterion to the case where $W = W_m^*$ and $G = G_m$. Let $w \neq 0$ be a point in W .

Theorem 3.2. *A point w as above is stable if and only if there exists a point w' in the orbit $G \cdot w$ of w such that $\lambda(\mathbb{C}^*) \cdot w'$ is closed in W for every special one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow G$ of G .*

Proof. We prove this by induction on $\dim(G \cdot w)$. If $\dim(G \cdot w) = 0$, the statement of the above theorem is obviously true. Hence, fixing a positive integer k , assume that the statement is true for all $0 \neq w \in W$ such that $\dim(G \cdot w) < k$. Now, let $0 \neq w \in W$ be such that $\dim(G \cdot w) = k$, and the proof is reduced to showing the statement for such a point w . Let $\Sigma(G)$ be the set of all special one-parameter subgroups of G . Fix a G_c -invariant Hermitian metric $\| \cdot \|$ on W . The proof is divided into three steps:

Step 1: First, we prove “only if” part of Theorem 3.2. Assume that w is stable. Since $G \cdot w$ is closed in W , the nonnegative function on $G \cdot w$ defined by

$$(3.3) \quad G \cdot w \ni g \cdot w \mapsto \|g \cdot w\| \in \mathbb{R}, \quad g \in G,$$

has a critical point at some point w' in $G \cdot w$. Let $\lambda \in \Sigma(G)$, and it suffices to show the closedness of $\lambda(\mathbb{C}^*) \cdot w'$ in W . We may assume that $\dim \lambda(\mathbb{C}^*) \cdot w' > 0$. Then by using the coordinate system associated to an orthonormal basis for W , we can write w' as $(w'_0, \dots, w'_r, 0, \dots, 0)$ in such a way that $w'_\alpha \neq 0$ for all $0 \leq \alpha \leq r$ and that

$$\lambda(e^t) \cdot w' = (e^{t\gamma_0} w'_0, \dots, e^{t\gamma_r} w'_r, 0, \dots, 0), \quad t \in \mathbb{C},$$

where γ_α , $\alpha = 0, 1, \dots, r$, are integers independent of the choice of t in \mathbb{C} . Since the closed orbit $G \cdot w$ does not contain the origin of W , the inclusion $\lambda(\mathbb{C}^*) \cdot w' \subset G \cdot w$ shows that $r \geq 1$ and that the coincidence $\gamma_0 = \gamma_1 = \dots = \gamma_r$ cannot occur. In particular,

$$f(t) := \log \|\lambda(e^t) \cdot w'\|^2 = \log (e^{2t\gamma_0} |w'_0|^2 + e^{2t\gamma_1} |w'_1|^2 + \dots + e^{2t\gamma_r} |w'_r|^2), \quad t \in \mathbb{R},$$

satisfies $f''(t) > 0$ for all t . Moreover, since the function in (3.3) has a critical point at w' , we have $f'(0) = 0$. It now follows that $\lim_{t \rightarrow +\infty} f(t) = +\infty$ and $\lim_{t \rightarrow -\infty} f(t) = +\infty$. Hence $\lambda(\mathbb{C}^*) \cdot w'$ is closed in W , as required.

Step 2: To prove “if” part of Theorem 3.2, we may assume that $w = w'$ without loss of generality. Hence, suppose that $\lambda(\mathbb{C}^*) \cdot w$ is closed in W for every $\lambda \in \Sigma(G)$. It then suffices to show that $G \cdot w$ is closed in W . For contradiction, assume that $G \cdot w$ is not closed in W . Since the closure of $G \cdot w$ in W always contains a closed orbit O_1 in W ,

by $\dim O_1 < \dim(G \cdot w) = k$, the induction hypothesis shows that there exists a point $\hat{w} \in O_1$ such that

$$(3.4) \quad \lambda(\mathbb{C}^*) \cdot \hat{w} \text{ is closed in } W \text{ for every } \lambda \in \Sigma(G).$$

Moreover, there exist elements g_i , $i = 1, 2, \dots$, in G such that $g_i \cdot w$ converges to \hat{w} in W . Then for each i , we can write $g_i = \kappa'_i \cdot \exp(2\pi A_i) \cdot \kappa_i$ for some $\kappa_i, \kappa'_i \in G_c$ and for some $A_i \in \mathfrak{a}$, where $2\pi\sqrt{-1}\mathfrak{a}$ is the Lie algebra of some maximal compact torus in G_c . Let $2\pi\sqrt{-1}\mathfrak{a}_{\mathbb{Z}}$ be the kernel of the exponential map of the Lie algebra $2\pi\sqrt{-1}\mathfrak{a}$, and put $\mathfrak{a}_{\mathbb{Q}} := \mathfrak{a}_{\mathbb{Z}} \otimes \mathbb{Q}$. Replacing $\{\kappa_i\}$ by its subsequence if necessary, we may assume that

$$(3.5) \quad \kappa_i \rightarrow \kappa_{\infty} \text{ and } \{\exp(2\pi A_i) \cdot \kappa_i\} \cdot w \rightarrow w_{\infty}, \quad \text{as } i \rightarrow \infty,$$

for some $\kappa_{\infty} \in G_c$ and $w_{\infty} \in G_c \cdot \hat{w}$. Then by (3.4), the orbit $\lambda(\mathbb{C}^*) \cdot w_{\infty}$ is also closed in W for every $\lambda \in \Sigma(G)$. Let \mathfrak{a}_{∞} denote the Lie subalgebra of \mathfrak{a} consisting of all elements in \mathfrak{a} whose associated vector fields on W vanish at $\kappa_{\infty} \cdot w$. For a Euclidean metric on \mathfrak{a} induced from a suitable bilinear form on $\mathfrak{a}_{\mathbb{Q}}$ defined over \mathbb{Q} , we write \mathfrak{a} as a direct sum $\mathfrak{a}_{\infty}^{\perp} \oplus \mathfrak{a}_{\infty}$, where $\mathfrak{a}_{\infty}^{\perp}$ is the orthogonal complement of \mathfrak{a}_{∞} in \mathfrak{a} . Let \bar{A}_i be the image of A_i under the orthogonal projection

$$\text{pr}_1 : \mathfrak{a} (= \mathfrak{a}_{\infty}^{\perp} \oplus \mathfrak{a}_{\infty}) \rightarrow \mathfrak{a}_{\infty}^{\perp}, \quad A \mapsto \bar{A} := \text{pr}_1(A).$$

Note that $\{\exp(2\pi A_i) \cdot \kappa_{\infty}\} \cdot w = \{\exp(2\pi \bar{A}_i) \cdot \kappa_{\infty}\} \cdot w$. Hence,

$$(3.6) \quad \begin{aligned} \limsup_{i \rightarrow \infty} \|\exp\{2\pi \text{Ad}(\kappa_{\infty}^{-1}) \bar{A}_i\} \cdot w\| &= \limsup_{i \rightarrow \infty} \|\{\exp(2\pi A_i) \cdot \kappa_{\infty}\} \cdot w\| \\ &\leq \lim_{i \rightarrow \infty} \|\{\exp(2\pi A_i) \cdot \kappa_i\} \cdot w\| = \|w_{\infty}\| < +\infty. \end{aligned}$$

Step 3: Since $\lambda(\mathbb{C}^*) \cdot w$ is closed in W for every $\lambda \in \Sigma(G)$, by the boundedness in (3.6), $\{\bar{A}_i\}$ is a bounded sequence in $\mathfrak{a}_{\infty}^{\perp}$ (see Remark 3.7 below). Hence, for some element A_{∞} in $\mathfrak{a}_{\infty}^{\perp}$, replacing $\{\bar{A}_i\}$ by its subsequence if necessary, we may assume that $\bar{A}_i \rightarrow A_{\infty}$ as $i \rightarrow \infty$. Then by (3.5),

$$w_{\infty} = \lim_{i \rightarrow \infty} \{\exp(2\pi \bar{A}_i) \cdot \kappa_i\} \cdot w = \{\exp(2\pi \bar{A}_{\infty}) \cdot \kappa_{\infty}\} \cdot w.$$

Since we have $\exp(2\pi \bar{A}_{\infty}) \in G$, the point w_{∞} in O_1 belongs to the orbit $G \cdot w$. This contradicts $O_1 \cap (G \cdot w) = \emptyset$, as required. The proof of Lemma 3.2 is now complete.

Remark 3.7. The boundedness of the sequence $\{\bar{A}_i\}$ in $\mathfrak{a}_{\infty}^{\perp}$ in Step 3 above can be seen as follows: For contradiction, we assume that the sequence $\{\bar{A}_i\}$ is unbounded. Put $v := \kappa_{\infty} \cdot w$ for simplicity. Then by (3.6), we first observe that

$$(3.8) \quad \limsup_{i \rightarrow \infty} \|\exp(2\pi \bar{A}_i) \cdot v\| < +\infty.$$

Since $2\pi\sqrt{-1}\mathfrak{a}_{\infty}$ is the Lie algebra of the isotropy subgroup of the compact torus $\exp(2\pi\sqrt{-1}\mathfrak{a})$ at v , both \mathfrak{a}_{∞} and $\mathfrak{a}_{\infty}^{\perp}$ are defined over \mathbb{Q} in \mathfrak{a} . By choosing a complex coordinate system

of W , we can write v as $(v_0, \dots, v_r, 0, \dots, 0)$ for some integer r with $0 \leq r \leq \dim W - 1$ such that $v_\alpha \neq 0$ for all $0 \leq \alpha \leq r$ and that

$$(3.9) \quad \exp(2\pi\bar{A}) \cdot v = (e^{2\pi\chi_0(\bar{A})}v_0, \dots, e^{2\pi\chi_r(\bar{A})}v_r, 0, \dots, 0), \quad \bar{A} \in \mathfrak{a}_\infty^\perp,$$

where $\chi_\alpha : \mathfrak{a}_\infty^\perp \rightarrow \mathbb{R}$, $\alpha = 0, 1, \dots, r$, are additive characters defined over \mathbb{Q} . Put $n := \dim_{\mathbb{R}} \mathfrak{a}_\infty^\perp$, and let $(\mathfrak{a}_\infty^\perp)_{\mathbb{Q}}$ denote the set of all rational points in $\mathfrak{a}_\infty^\perp$. Let us now identify

$$\mathfrak{a}_\infty^\perp = \mathbb{R}^n \quad \text{and} \quad (\mathfrak{a}_\infty^\perp)_{\mathbb{Q}} = \mathbb{Q}^n,$$

as vector spaces. Since the orbit $\lambda(\mathbb{C}^*) \cdot w$ is closed in W for all special one-parameter subgroups $\lambda : \mathbb{C}^* \rightarrow G$ of G , the same thing is true also for $\lambda(\mathbb{C}^*) \cdot v$. Hence,

$$(3.10) \quad \mathbb{Q}^n \setminus \{0\} \subset \bigcup_{\alpha, \beta=0}^r U_{\alpha\beta},$$

where $U_{\alpha\beta} := \{A \in \mathfrak{a} ; \chi_\alpha(A) > 0 > \chi_\beta(A)\}$. Note that the boundaries of the open sets $U_{\alpha\beta}$, $1 \leq \alpha \leq r$, $1 \leq \beta \leq r$, in \mathbb{R}^n sit in the union of \mathbb{Q} -hyperplanes

$$H_\alpha := \{\chi_\alpha = 0\}, \quad \alpha = 0, 1, \dots, r,$$

in \mathbb{R}^n . Since an intersection of any finite number of hyperplanes H_α , $\alpha = 0, 1, \dots, r$, has dense rational points, (3.10) above easily implies

$$(3.11) \quad \mathbb{R}^n \setminus \{0\} = \bigcup_{\alpha, \beta=0}^r U_{\alpha\beta}.$$

Replacing $\{\bar{A}_i\}$ by its suitable subsequence if necessary, we may assume that there exists an element A_∞ in $\mathfrak{a}_\infty^\perp (= \mathbb{R}^n)$ with $\|A_\infty\|_{\mathfrak{a}} = 1$ such that

$$\lim_{i \rightarrow \infty} \frac{\bar{A}_i}{\|\bar{A}_i\|_{\mathfrak{a}}} = A_\infty,$$

where $\|\cdot\|_{\mathfrak{a}}$ denotes the Euclidean norm for \mathfrak{a} as in Step 2 in the proof of Theorem 3.2. By (3.11), there exist $\alpha, \beta \in \{0, 1, \dots, r\}$ such that $A_\infty \in U_{\alpha\beta}$, and in particular $\chi_\alpha(A_\infty) > 0$. On the other hand, $\limsup_{i \rightarrow \infty} \|\bar{A}_i\|_{\mathfrak{a}} = +\infty$ by our assumption. Thus,

$$\limsup_{i \rightarrow \infty} \chi_\alpha(\bar{A}_i) = \limsup_{i \rightarrow \infty} \{\|\bar{A}_i\|_{\mathfrak{a}} \cdot \chi_\alpha(\bar{A}_i / \|\bar{A}_i\|_{\mathfrak{a}})\} = (\limsup_{i \rightarrow \infty} \|\bar{A}_i\|_{\mathfrak{a}}) \chi_\alpha(A_\infty) = +\infty,$$

in contradiction to (3.8) and (3.9), as required.

4. THE CHOW NORM

Take an algebraic torus $T \subset \text{Aut}^0(M)$, and let $\iota : \text{SL}(V_m) \rightarrow \text{PGL}(V_m)$ be the natural projection, where we regard $\text{Aut}^0(M)$ as a subgroup of $\text{PGL}(V_m)$ via the Kodaira embedding $\Phi_m : M \hookrightarrow \mathbb{P}^*(V_m)$, $m \gg 1$. In this section, we fix a \tilde{T}_c -invariant Hermitian metric ρ on V_m , where \tilde{T}_c is the maximal compact subgroup of $\tilde{T} := \iota^{-1}(T)$. Obviously, in terms of this metric, $V(\chi_k) \perp V(\chi_\ell)$ if $k \neq \ell$. Using Deligne's pairings (cf. [8], 8.3), Zhang

([39], 1.5) defined a special type of norm on W_m^* , called the *Chow norm*, as a nonnegative real-valued function

$$(4.1) \quad W_m^* \ni w \longmapsto \|w\|_{\text{CH}(\rho)} \in \mathbb{R}_{\geq 0},$$

with very significant properties described below. First, this is a norm, so that it has the only zero at the origin satisfying the homogeneity condition

$$\|cw\|_{\text{CH}(\rho)} = |c| \cdot \|w\|_{\text{CH}(\rho)} \quad \text{for all } (c, w) \in \mathbb{C} \times W_m^*.$$

For the group $\text{SL}(V_m)$, we consider the maximal compact subgroup $\text{SU}(V_m; \rho)$. For a special one-parameter subgroup

$$\lambda : \mathbb{C}^* \rightarrow \text{SL}(V_m)$$

of $\text{SL}(V_m)$, there exist integers γ_j , $j = 0, 1, \dots, N_m$, and an orthonormal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for (V_m, ρ) such that, for all j ,

$$(4.2) \quad \lambda_z \cdot s_j = e^{z\gamma_j} s_j, \quad z \in \mathbb{C},$$

where $\lambda_z := \lambda(e^z)$. Recall that the subvariety M_m in $\mathbb{P}^*(V_m)$ is the image of the Kodaira embedding $\Phi_m : M \hookrightarrow \mathbb{P}^*(V_m)$ defined by

$$(4.3) \quad \Phi_m(p) = (s_0(p) : s_1(p) : \dots : s_{N_m}(p)), \quad p \in M,$$

where $\mathbb{P}^*(V_m)$ is identified with $\mathbb{P}^{N_m}(\mathbb{C}) = \{(z_0 : z_1 : \dots : z_{N_m})\}$. Put $M_{m,t} := \lambda_t(M_m)$ for each $t \in \mathbb{R}$. As in Section 2, $\hat{M}_{m,t} := \lambda_t \cdot \hat{M}_m$ is the nonzero point of W_m^* sitting over the Chow point of the irreducible reduced cycle $M_{m,t}$ on $\mathbb{P}^*(V_m)$. Then (cf. [39], 1.4, 3.4.1)

$$(4.4) \quad \frac{d}{dt} \left(\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)} \right) = (n+1) \int_M \frac{\sum_{j=0}^{N_m} \gamma_j |\lambda_t \cdot s_j|^2}{\sum_{j=0}^{N_m} |\lambda_t \cdot s_j|^2} (\Phi_m^* \lambda_t^* \omega_{\text{FS}})^n,$$

where ω_{FS} is the Fubini-Study form $(\sqrt{-1}/2\pi) \partial \bar{\partial} \log(\sum_{j=0}^{N_m} |z_j|^2)$ on $\mathbb{P}^*(V_m)$, and we regard λ_t as a linear transformation of $\mathbb{P}^*(V_m)$ induced by (4.2). Note that the term $\Phi_m^* \lambda_t^* \omega_{\text{FS}}$ above is just $(\sqrt{-1}/2\pi) \partial \bar{\partial} \log(\sum_{j=0}^{N_m} |\lambda_t \cdot s_j|^2)$. Put $\Gamma := 2\pi\sqrt{-1}\mathbb{Z}$. By setting

$$\mathbb{C}/\Gamma = \{t + \sqrt{-1}\theta; t \in \mathbb{R}, \theta \in \mathbb{R}/(2\pi\mathbb{Z})\},$$

we consider the complexified situation. Let $\eta : M \times \mathbb{C}/\Gamma \rightarrow \mathbb{P}^*(V_m)$ be the map sending each $(p, t + \sqrt{-1}\theta)$ in $M \times \mathbb{C}/\Gamma$ to $\lambda_{t+\sqrt{-1}\theta} \cdot \Phi_m(p)$ in $\mathbb{P}^*(V_m)$. For simplicity, we put

$$Q := \frac{\sum_{j=0}^{N_m} \gamma_j e^{2t\gamma_j} |s_j|^2}{\sum_{j=0}^{N_m} e^{2t\gamma_j} |s_j|^2} \left(= \frac{\sum_{j=0}^{N_m} \gamma_j |\lambda_t \cdot s_j|^2}{\sum_{j=0}^{N_m} |\lambda_t \cdot s_j|^2} \right).$$

We further put $z := t + \sqrt{-1}\theta$. For the time being, on the total complex manifold $M \times \mathbb{C}/\Gamma$, the ∂ -operator and the $\bar{\partial}$ -operator will be written simply as ∂ and $\bar{\partial}$ respectively, while on M , they will be denoted by ∂_M and $\bar{\partial}_M$ respectively. Then

$$\eta^* \omega_{\text{FS}} = \Phi_m^* \lambda_t^* \omega_{\text{FS}} + \frac{\sqrt{-1}}{2\pi} (\partial_M Q \wedge d\bar{z} + dz \wedge \bar{\partial}_M Q) + \frac{\sqrt{-1}}{4\pi} \frac{\partial Q}{\partial t} dz \wedge d\bar{z}.$$

For $0 \neq r \in \mathbb{R}$, we consider the 1-chain $I_r := [0, r]$, where $[0, r]$ means the 1-chain $-[r, 0]$ if $r < 0$. Let $\text{pr} : \mathbb{C}/\Gamma \rightarrow \mathbb{R}$ be the mapping sending each $t + \sqrt{-1}\theta$ to t . We now put $B_r := \text{pr}^* I_r$. Then $\int_{M \times B_r} \eta^* \omega_{\text{FS}}^{n+1}$ is nothing but

$$\begin{aligned} & (n+1) \int_0^r dt \int_M \left(\frac{\partial Q}{\partial t} \Phi_m^* \lambda_t^* \omega_{\text{FS}}^n + \frac{\sqrt{-1}}{\pi} \bar{\partial}_M Q \wedge \partial_M Q \wedge n \Phi_m^* \lambda_t^* \omega_{\text{FS}}^{n-1} \right) \\ &= \int_0^r \frac{d^2}{dt^2} \left(\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)} \right) dt = \frac{d}{dt} \left(\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)} \right) \Big|_{t=0}^{t=r}, \end{aligned}$$

and by assuming $r \geq 0$, we obtain the following convexity formula:

Theorem 4.5.
$$\frac{d}{dt} \left(\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)} \right) \Big|_{t=0}^{t=r} = \int_{M \times B_r} \eta^* \omega_{\text{FS}}^{n+1} \geq 0.$$

Remark 4.6. Besides special one-parameter subgroups of $\text{SL}(V_m)$, we also consider a little more general smooth path λ_t , $t \in \mathbb{R}$, in $\text{GL}(V_m)$ written explicitly by

$$\lambda_t \cdot s_j = e^{t\gamma_j + \delta_j} s_j, \quad j = 0, 1, \dots, N_m,$$

where $\gamma_j, \delta_j \in \mathbb{R}$ are not necessarily rational. In this case also, we easily see that the formula (4.4) and Theorem 4.5 are still valid.

5. PROOF OF THEOREM A

The statement of Theorem A is divided into “if” part, “only if” part, and the uniqueness part. We shall prove these three parts separately.

Proof of “if” part. Let $\omega \in \mathcal{S}$ be a critical metric relative to T . Then by Definition 2.6, in terms of the Hermitian metric defined in (2.4), there exists an admissible normal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for V_m of index b such that the associated function $E_{\omega,b}$ has a constant value C on M . By operating $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log$ on the identity $E_{\omega,b} = C$, we have

$$(5.1) \quad \Phi_m^* \omega_{\text{FS}} = m \omega.$$

Besides the Hermitian metric defined in (2.4), we shall now define another Hermitian metric on V_m . By the identification $V_m \cong \mathbb{C}^{N_m}$ via the basis $\{s_0, s_1, \dots, s_{N_m}\}$, the standard Hermitian metric on \mathbb{C}^{N_m} induces a Hermitian metric ρ on V_m . As a maximal compact subgroup of G_m , we choose $(G_m)_c$ as in Section 2 by using the metric defined in (2.4). Then the Hermitian metric ρ is also preserved by the $(G_m)_c$ -action on V_m . Let

$$\lambda : \mathbb{C}^* \rightarrow G_m$$

be a special one-parameter subgroup of G_m . By the notation $l(k, i)$ as in Definition 2.3, we put $s_{k,i} := s_{l(k,i)}$. If necessary, replacing $\{s_0, s_1, \dots, s_{N_m}\}$ by another admissible normal basis for V_m of the same index b , we may assume without loss of generality that there exist integers $\gamma_{k,i}$, $i = 1, 2, \dots, n_k$, satisfying

$$(5.2) \quad \lambda_t \cdot s_{k,i} = e^{t\gamma_{k,i}} s_{k,i}, \quad t \in \mathbb{C},$$

where $\lambda_t := \lambda(e^t)$ is as in (4.2), and the equality $\sum_{i=1}^{n_k} \gamma_{k,i} = 0$ is required to hold for every k . Put $\gamma_{k,i} = \gamma_{l(k,i)}$ for simplicity. Then by (4.4) and (5.1),

$$\begin{aligned} \frac{d}{dt} \left(\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)} \right)_{|t=0} &= (n+1) \int_M \frac{\sum_{j=0}^{N_m} \gamma_j |s_j|^2}{\sum_{j=0}^{N_m} |s_j|^2} (\Phi_m^* \omega_{\text{FS}})^n \\ &= (n+1) m^n \int_M \frac{\sum_{j=0}^{N_m} \gamma_j |s_j|_{h^m}^2}{\sum_{j=0}^{N_m} |s_j|_{h^m}^2} \omega^n = (n+1) m^n \int_M \frac{\sum_{k=1}^{\nu_m} (\sum_{i=1}^{n_k} \gamma_{k,i} |s_i|_{h^m}^2)}{E_{\omega,b}} \omega^n \\ &= \frac{(n+1) m^n}{C} \int_M \sum_{k=1}^{\nu_m} (\sum_{i=1}^{n_k} \gamma_{k,i} |s_i|_{h^m}^2) \omega^n = \frac{(n+1) m^n}{C} \sum_{k=1}^{\nu_m} b_k (\sum_{i=1}^{n_k} \gamma_{k,i}) = 0. \end{aligned}$$

Note also that, by Theorem 4.5, we have $c := (d^2/dt^2)(\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)})_{|t=0} \geq 0$.

Case 1: If c is positive, then $\lim_{t \rightarrow -\infty} \|\hat{M}_{m,t}\|_{\text{CH}(\rho)} = +\infty = \lim_{t \rightarrow +\infty} \|\hat{M}_{m,t}\|_{\text{CH}(\rho)}$, and in particular $\lambda(\mathbb{C}^*) \cdot \hat{M}_m$ is closed.

Case 2: If c is zero, then by applying Theorem 4.5 infinitesimally, we see that $\lambda(\mathbb{C}^*)$ preserves the subvariety M_m in $\mathbb{P}^*(V_m)$, and moreover by $(d/dt)(\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)})_{|t=0} = 0$, the isotropy representation of $\lambda(\mathbb{C}^*)$ on the complex line $\mathbb{C}\hat{M}_m$ is trivial. Hence, $\lambda(\mathbb{C}^*) \cdot \hat{M}_m$ is a single point, and in particular closed.

Thus, these two cases together with Theorem 3.2 show that the subvariety M_m of $\mathbb{P}^*(V_m)$ is stable relative to T , as required.

Remark 5.3. About the one-parameter subgroup $\{\lambda_t; t \in \mathbb{R}\}$ of G_m , we consider a more general situation that $\gamma_{k,i}$ in (5.2) are just real numbers which are not necessarily rational. The above computation together with Remark 4.6 shows that, even in this case, $(d/dt)_{t=0}(\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)})$ vanishes.

Proof of “only if” part. Assume that the subvariety M_m in $\mathbb{P}^*(V_m)$ is stable relative to T . Take a Hermitian metric ρ for V_m such that $V(\chi_k) \perp V(\chi_\ell)$ for $k \neq \ell$. For this ρ , we consider the associated Chow norm. Since the orbit $G_m \cdot \hat{M}_m$ is closed in W_m , the Chow norm restricted to this orbit attains an absolute minimum. Hence, for some $g_0 \in G_m$,

$$0 \neq \|g_0 \cdot \hat{M}_m\|_{\text{CH}(\rho)} \leq \|g \cdot \hat{M}_m\|_{\text{CH}(\rho)}, \quad \text{for all } g \in G_m.$$

By choosing an admissible normal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for $(V_m; \rho)$ of index $(1, 1, \dots, 1)$, we identify V_m with $\mathbb{C}^{N_m} = \{(z_0, z_1, \dots, z_{N_m})\}$. Then $\text{SL}(V_m)$ is identified with $\text{SL}(N_m + 1; \mathbb{C})$. Let \mathfrak{g}_m be the Lie subalgebra of $\mathfrak{sl}(N_m + 1; \mathbb{C})$ associated to the Lie subgroup G_m of $\text{SL}(N_m + 1; \mathbb{C})$. We can now write $g_0 = \kappa' \cdot \exp\{\text{Ad}(\kappa)D\}$ for some $\kappa, \kappa' \in G_{m,c}$ and a real diagonal matrix D in \mathfrak{g}_m . By $\|\exp\{\text{Ad}(\kappa)D\} \cdot \hat{M}_m\|_{\text{CH}(\rho)} = \|g_0 \cdot \hat{M}_m\|_{\text{CH}(\rho)}$, we have

$$(5.4) \quad \|\exp\{\text{Ad}(\kappa)D\} \cdot \hat{M}_m\|_{\text{CH}(\rho)} \leq \|\exp\{t \text{Ad}(\kappa)A\} \cdot \exp\{\text{Ad}(\kappa)D\} \cdot \hat{M}_m\|_{\text{CH}(\rho)}, \quad t \in \mathbb{R},$$

for every real diagonal matrix A in \mathfrak{g}_m . For $j = 0, 1, \dots, N_m$, we write the j -th diagonal element of A and D above as a_j and d_j , respectively. Put $c_j := \exp d_j$ and $s'_j := \kappa^{-1} \cdot s_j$.

Then $\{s'_0, s'_1, \dots, s'_{N_m}\}$ is again an admissible normal basis for (V_m, ρ) of index $(1, 1, \dots, 1)$. By the notation in Definition 2.3, we rewrite s'_j, a_j, c_j, z_j as $s'_{k,i}, a_{k,i}, c_{k,i}, z_{k,i}$ by

$$s'_{k,i} := s'_{l(k,i)}, \quad a_{k,i} := a_{l(k,i)}, \quad c_{k,i} := c_{l(k,i)}, \quad z_{k,i} := z_{l(k,i)},$$

where $k = 1, 2, \dots, \nu_m$ and $i = 1, 2, \dots, n_k$. By (5.4), the derivative at $t = 0$ of the right-hand side of (5.4) vanishes. Hence by (4.4) together with Remark 4.6, fixing an arbitrary real diagonal matrix A in \mathfrak{g}_m , we have

$$(5.5) \quad \int_M \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} a_{k,i} c_{k,i}^2 |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2} \Phi_m^*(\Theta^n) = 0$$

where we set $\Theta := (\sqrt{-1}/2\pi) \partial \bar{\partial} \log(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |z_{k,i}|^2)$. Let $k_0 \in \{1, 2, \dots, \nu_m\}$ and let $i_1, i_2 \in \{1, 2, \dots, n_k\}$ with $i_1 \neq i_2$. Using Kronecker's delta, we specify the real diagonal matrix A by setting

$$a_{k,i} = \delta_{kk_0} (\delta_{ii_1} - \delta_{ii_2}), \quad k = 1, 2, \dots, \nu_m; \quad i = 1, 2, \dots, n_k.$$

Apply (5.5) to this A , and let (i_1, i_2) run through the set of all pairs of two distinct elements in $\{1, 2, \dots, n_k\}$. Then there exists a positive constant $b_k > 0$ independent of the choice of i in $\{1, 2, \dots, n_k\}$ such that

$$(5.6) \quad \frac{N_m + 1}{m^n c_1(L)^n [M]} \int_M \frac{c_{k,i}^2 |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2} \Phi_m^*(\Theta^n) = b_k, \quad k = 1, 2, \dots, \nu_m.$$

The following identity (5.7) allows us to define (cf. [39]) a Hermitian metric h_{FS} on L^m by

$$(5.7) \quad |s|_{h_{\text{FS}}}^2 := \frac{(N_m + 1)}{c_1(L)^n [M]} \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |(s, s'_{k,i})_\rho|^2 |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2}, \quad s \in V_m.$$

Then for this Hermitian metric, it is easily seen that

$$(5.8) \quad \sum_{j=0}^{N_m} |c_j s'_j|_{h_{\text{FS}}}^2 = \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|_{h_{\text{FS}}}^2 = (N_m + 1)/c_1(L)^n [M].$$

By operating $(\sqrt{-1}/2\pi) \partial \bar{\partial} \log$ on both sides of (5.8), we obtain $\Phi_m^* \Theta = c_1(L^m; h_{\text{FS}})$. We now set $h := (h_{\text{FS}})^{1/m}$ and $\omega := c_1(L; h)$. Then

$$\omega = (1/m) \Phi_m^* \Theta.$$

Put $s''_{k,i} := c_{k,i} s'_{k,i}$, and as in Definition 2.3, we write $s''_{k,i}$ as $s''_{l(k,i)}$. Then by (5.8), we have the equality $\sum_{j=0}^{N_m} |s''_j|_{h^m}^2 = (N_m + 1)/c_1(L)^n [M]$. Moreover, in terms of the Hermitian metric defined in (2.4), the equality (5.6) is interpreted as

$$\|s''_{k,i}\|_{L^2}^2 = b_k, \quad k = 1, 2, \dots, \nu_m; \quad i = 1, 2, \dots, n_k,$$

while by this together with (5.8) above, we obtain $\sum_{k=1}^{\nu_m} n_k b_k = N_m + 1$, as required.

Proof of *uniqueness*. Let $\omega = c_1(L; h)$ and $\omega' = c_1(L; h')$ be critical metrics relative to T , and let $\{s_j; j = 0, 1, \dots, N_m\}$ and $\{s'_j; j = 0, 1, \dots, N_m\}$ be respectively the associated admissible normal bases for V_m of index b . We use the notation in Definition 2.3. Then

$$E_{\omega, b} := \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s_{k,i}|_{h^m}^2 \quad \text{and} \quad E_{\omega', b} := \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s'_{k,i}|_{h'^m}^2$$

take the same constant value $C := (N_m + 1)/c_1(L)^n [M]$ on M . Note here that, by operating $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log$ on both of these identities, we obtain

$$m\omega = (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s_{k,i}|^2) \quad \text{and} \quad m\omega' = (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s'_{k,i}|^2).$$

If necessary, we replace each $s_{k,i}$ by $\zeta_k s_{k,i}$ for a suitable complex number ζ_k , independent of i , of absolute value 1. Then for each $k = 1, 2, \dots, \nu_m$, we may assume that there exist a matrix $g^{(k)} = (g_{i\hat{i}}^{(k)}) \in \text{GL}(n_k; \mathbb{C})$ satisfying

$$s'_{k,\hat{i}} = \sum_{i=1}^{n_k} s_{k,i} g_{i\hat{i}}^{(k)},$$

where i and \hat{i} always run through the integers in $\{1, 2, \dots, n_k\}$. Then the matrix $g^{(k)}$ above is written as $\kappa^{(k)} \cdot (\exp A^{(k)}) \cdot (\kappa'^{(k)})^{-1}$ for some real diagonal matrix $A^{(k)}$ and

$$\kappa^{(k)} = (\kappa_{i\hat{i}}^{(k)}) \quad \text{and} \quad \kappa'^{(k)} = (\kappa'_{i\hat{i}}^{(k)})$$

in $\text{SU}(n_k)$. Let $a_i^{(k)}$ be the i -th diagonal element of $A^{(k)}$. For each \hat{i} , we put $\tilde{s}_{k,\hat{i}} := \sum_{i=1}^{n_k} s_{k,i} \kappa_{i\hat{i}}^{(k)}$ and $\tilde{s}'_{k,\hat{i}} := \sum_{i=1}^{n_k} s'_{k,i} \kappa'_{i\hat{i}}^{(k)}$. If necessary, we replace the bases $\{s_{k,1}, s_{k,2}, \dots, s_{k,n_k}\}$ and $\{s'_{k,1}, s'_{k,2}, \dots, s'_{k,n_k}\}$ for $V(\chi_k)$ by the bases $\{\tilde{s}_{k,1}, \tilde{s}_{k,2}, \dots, \tilde{s}_{k,n_k}\}$ and $\{\tilde{s}'_{k,1}, \tilde{s}'_{k,2}, \dots, \tilde{s}'_{k,n_k}\}$, respectively. Then we may assume, from the beginning, that

$$s'_{k,i} = \{\exp a_i^{(k)}\} s_{k,i}, \quad i = 1, 2, \dots, n_k.$$

We now set $\tau_{k,i} := s_{k,i}/\sqrt{b_k}$, and the Hermitian metric for V_m defined in (2.4) will be denoted by ρ . Then $\{\tau_{k,i}; k = 1, 2, \dots, \nu_m, i = 1, 2, \dots, n_k\}$ is an admissible normal basis of index $(1, 1, \dots, 1)$ for (V_m, ρ) . Let $\{\lambda_t; t \in \mathbb{C}\}$ be the smooth one-parameter family of elements in $\text{GL}(V_m)$ defined by

$$\lambda_t \cdot \tau_{k,i} = \{\exp(t a_i^{(k)})\} \sqrt{b_k} \tau_{k,i}, \quad k = 1, 2, \dots, \nu_m; i = 1, 2, \dots, n_k.$$

Put $\hat{M}_{m,t} := \lambda_t \cdot \hat{M}_m$, $0 \leq t \leq 1$. Then by Remark 4.6 applied to the formula (4.4), the derivative $\mathfrak{d}(t) := (d/dt)(\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)})/(n+1)$ at $t \in [0, 1]$ is expressible as

$$\int_M \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} a_i^{(k)} |\lambda_t \cdot \tau_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\lambda_t \cdot \tau_{k,i}|^2} \{(\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\lambda_t \cdot \tau_{k,i}|^2)\}^n$$

Hence at $t = 0$, we see that

$$\mathfrak{d}(0) = \int_M \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \{a_i^{(k)} |s_{k,i}|_{h^m}^2 / C\} (m\omega)^n = (m^n / C) \sum_{k=1}^{\nu_m} \{b_k \sum_{i=1}^{n_k} a_i^{(k)}\},$$

while at $t = 1$ also, we obtain

$$\mathfrak{d}(1) = \int_M \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \{a_i^{(k)} |s'_{k,i}|_{h'^m}^2 / C\} (m\omega')^n = (m^n / C) \sum_{k=1}^{\nu_m} \{b_k \sum_{i=1}^{n_k} a_i^{(k)}\}.$$

Thus, $\mathfrak{d}(0)$ coincides with $\mathfrak{d}(1)$, while by Remark 4.6, we see from Theorem 4.5 that $(d^2/dt^2)\{\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)}\} \geq 0$ on $[0, 1]$. Hence, for all $t \in [0, 1]$,

$$\frac{d^2}{dt^2} \{\log \|\hat{M}_{m,t}\|_{\text{CH}(\rho)}\} = 0, \quad \text{on } M.$$

By Remark 4.6, the formula in Theorem 4.5 shows that λ_t , $t \in [0, 1]$, belong to H up to a positive scalar multiple. Since λ_1 commutes with T , the uniqueness follows, as required.

6. PROOF OF THEOREM B

Throughout this section, we assume that the first Chern class $c_1(L)_{\mathbb{R}}$ admits an extremal Kähler metric $\omega_0 = c_1(L; h_0)$. Then by a theorem of Calabi [3], the identity component K of the group of isometries of (M, ω_0) is a maximal compact connected subgroup of H , and we obtain $\omega_0 \in \mathcal{S}_K$ by the notation in the introduction.

Definition 6.1. For a K -invariant Kähler metric $\omega \in \mathcal{S}_K$ on M in the class $c_1(L)_{\mathbb{R}}$, we choose a Hermitian metric h on L such that $\omega = c_1(L; h)$. Then the power series in q given by the right-hand side of (2.8) will be denoted by $\Psi(\omega, q)$. Given ω and q , the power series $\Psi(\omega, q)$ is independent of the choice of h .

Let \mathcal{D}_0 be the Lichnérowicz operator as defined in [3], (2.1), for the extremal Kähler manifold (M, ω_0) . Then by $\mathcal{V} \in \mathfrak{k}$, the operator \mathcal{D}_0 preserves the space \mathcal{F} of all real-valued smooth K -invariant functions φ such that $\int_M \varphi \omega_0^n = 0$. Hence, we regard \mathcal{D}_0 just as an operator $\mathcal{D}_0 : \mathcal{F} \rightarrow \mathcal{F}$, and the kernel in \mathcal{F} of this restricted operator will be denoted simply by $\text{Ker } \mathcal{D}_0$. Then $\text{Ker } \mathcal{D}_0$ is a subspace of \mathcal{K}_{ω_0} , and we have an isomorphism

$$(6.2) \quad e_0 : \text{Ker } \mathcal{D}_0 \cong \mathfrak{z}, \quad \varphi \leftrightarrow e_0(\varphi) := \text{grad}_{\omega_0}^{\mathbb{C}} \varphi.$$

By the inner product $(\cdot, \cdot)_{\omega_0}$ defined in the introduction, we write \mathcal{F} as an orthogonal direct sum $\text{Ker } \mathcal{D}_0 \oplus \text{Ker } \mathcal{D}_0^{\perp}$. We then consider the orthogonal projection

$$P : \mathcal{F} (= \text{Ker } \mathcal{D}_0 \oplus \text{Ker } \mathcal{D}_0^{\perp}) \rightarrow \text{Ker } \mathcal{D}_0.$$

Now, starting from $\omega(0) := \omega_0$, we inductively define a Hermitian metric $h(k)$, a Kähler metric $\omega(k) := c_1(L; h(k)) \in \mathcal{S}_K$, and a vector field $\mathcal{Y}(k) \in \sqrt{-1}\mathfrak{z}$, $k = 1, 2, \dots$, by

$$(6.3) \quad \begin{cases} h(k) := h(k-1) \exp(-q^k \varphi_k), \\ \omega(k) = \omega(k-1) + (\sqrt{-1}/2\pi) q^k \partial \bar{\partial} \varphi_k, \\ \mathcal{Y}(k) = \mathcal{Y}(k-1) + \sqrt{-1} q^{k+2} e_0(\zeta_k), \end{cases}$$

for appropriate $\varphi_k \in \text{Ker } \mathcal{D}_0^\perp$ and $\zeta_k \in \text{Ker } \mathcal{D}_0$, where $\omega(k)$ and $\mathcal{Y}(k)$ are required to satisfy the condition (2.8) with ℓ replaced by k . We now set $g(k) := \exp^\mathbb{C} \mathcal{Y}(k)$. Then

$$\begin{aligned} & \{h(k) \cdot g(k)\}^{-m} h(k)^m \{Z(q, \omega(k); \mathcal{Y}(k)) - C_q\} \\ &= \frac{n!}{m^n} \{\sum_{j=0}^N |s_j| h(k)^m\} - C_q \{g(k) \cdot h(k)^{-m}\} h(k)^m \\ &= \Psi(\omega(k), q) - C_q h(k)^m \{(\exp^\mathbb{C} \mathcal{Y}(k)) \cdot h(k)^{-m}\}, \\ &= \Psi(\omega(k), q) - C_q \{1 + h(k) (\mathcal{Y}(k)/q) \cdot h(k)^{-1} + R(\mathcal{Y}(k); h(k))\}, \end{aligned}$$

where $C_q = 1 + \sum_{k=0}^\infty \alpha_k q^{k+1}$ is a power series in q with real coefficients α_k specified later, and the last term $R(\mathcal{Y}(k); h(k)) := h(k)^m \sum_{j=2}^\infty \{\mathcal{Y}(k)^j / j!\} \cdot h(k)^{-m}$ will be taken care of as a higher order term in q . Consider the truncated term $C_{q,\ell} = 1 + \sum_{k=0}^\ell \alpha_k q^{k+1}$. Put

$$\Xi(\omega(k), \mathcal{Y}(k), C_{q,k}) := \Psi(\omega(k), q) - C_{q,k} \{1 - (\mathcal{Y}(k)/q) \cdot \log h(k) + R(\mathcal{Y}(k); h(k))\}$$

for each k . Then, in terms of $\omega(k)$, $\mathcal{Y}(k)$ and $C_{q,k}$, the condition (2.8) with ℓ replaced by k is just the equivalence

$$(6.4) \quad \Xi(\omega(k), \mathcal{Y}(k), C_{q,k}) \equiv 0, \quad \text{modulo } q^{k+2}.$$

We shall now define $\omega(k)$, $\mathcal{Y}(k)$ and $C_{q,k}$ inductively in such a way that the condition (6.4) is satisfied. If $k = 0$, then we set $\omega(0) = \omega_0$, $\mathcal{Y}(0) = \sqrt{-1} q^2 \mathcal{V}/2$ and $C_{q,0} = 1 + \alpha_0 q$, where we put $\alpha_0 := \{2c_1(L)^n [M]\}^{-1} \{\int_M \sigma_\omega \omega^n + 2\pi F(\mathcal{V})\}$ for $\omega \in \mathcal{S}_K$. This α_0 is obviously independent of the choice of ω in \mathcal{S}_K . Then, modulo q^2 ,

$$\begin{aligned} & \Psi(\omega(k), q) - C_{q,0} \{1 - (\mathcal{Y}(0)/q) \cdot \log h(0) + R(\mathcal{Y}(0); h(0))\} \\ & \equiv \left(1 + \frac{\sigma_{\omega_0}}{2} q\right) - (1 + \alpha_0 q) \{1 - q h_0^{-1} \sqrt{-1} (\mathcal{V}/2) \cdot h_0\} \\ & \equiv \left(1 + \frac{\sigma_{\omega_0}}{2} q\right) - (1 + \alpha_0 q) \left\{1 + \left(\frac{\sigma_{\omega_0}}{2} - \alpha_0\right) q\right\} \equiv 0, \end{aligned}$$

and we see that (6.4) is true for $k = 0$. Here, the equality $h_0^{-1} \sqrt{-1} (\mathcal{V}/2) \cdot h_0 = \alpha_0 - (\sigma_{\omega_0}/2)$ follows from a routine computation (see for instance [23]).

Hence, let $\ell \geq 1$ and assume (6.4) for $k = \ell - 1$. It then suffices to find φ_ℓ , ζ_ℓ and α_ℓ satisfying (6.4) for $k = \ell$. Put $\mathcal{Y}_\ell := \sqrt{-1} e_0(\zeta_\ell)$. For each $(\varphi_\ell, \zeta_\ell, \alpha_\ell) \in \text{Ker } \mathcal{D}_0^\perp \times \text{Ker } \mathcal{D}_0 \times \mathbb{R}$, we consider

$$\begin{aligned} \Phi(q; \varphi_\ell, \zeta_\ell, \alpha_\ell) &:= \Psi(\omega(\ell - 1) + (\sqrt{-1}/2\pi) q^\ell \partial \bar{\partial} \varphi_\ell, q) - \\ & (C_{q,\ell-1} + \alpha_\ell q^{\ell+1}) \left\{ 1 - (\mathcal{Y}(\ell - 1)/q + q^{\ell+1} \mathcal{Y}_\ell) \cdot \log \{h(\ell - 1) \exp(-q^\ell \varphi_\ell)\} \right. \\ & \quad \left. + R(\mathcal{Y}(\ell - 1)/q + q^{\ell+1} \mathcal{Y}_\ell; h(\ell - 1) \exp(-q^\ell \varphi_\ell)) \right\}. \end{aligned}$$

By the induction hypothesis, $\Xi(\omega(\ell-1), \mathcal{Y}(\ell-1), C_{q,\ell-1}) \equiv 0$ modulo $q^{\ell+1}$. Since $\Phi(q; 0, 0, 0) = \Xi(\omega(\ell-1), \mathcal{Y}(\ell-1), C_{q,\ell-1})$, we have

$$\Phi(q; 0, 0, 0) \equiv u_\ell q^{\ell+1}, \quad \text{modulo } q^{\ell+2},$$

for some real-valued K -invariant smooth function u_ℓ on M . Let $(\varphi_\ell, \zeta_\ell, \alpha_\ell) \in \text{Ker } \mathcal{D}_0^\perp \times \text{Ker } \mathcal{D}_0 \times \mathbb{R}$. Since φ_k is K -invariant, by $\mathcal{V} \in \mathfrak{k}$, we see that $\sqrt{-1}\mathcal{V}\varphi_k$ is a real-valued function on M . Note also that $\mathcal{Y}(0) = (\sqrt{-1}\mathcal{V}/2)q^2$. Then the variation formula for the scalar curvature (see for instance [3], (2.5)) shows that, modulo $q^{\ell+2}$,

$$\begin{aligned} & \Phi(q; \varphi_\ell, \zeta_\ell, \alpha_\ell) \\ & \equiv \Phi(q; 0, 0, 0) + \frac{q^{\ell+1}}{2} (-\mathcal{D}_0 + \sqrt{-1}\mathcal{V})\varphi_\ell - \alpha_\ell q^{\ell+1} + q^{\ell+1} h_0^{-1}(\mathcal{Y}_\ell \cdot h_0) - \frac{\sqrt{-1}}{2} \mathcal{V} \varphi_\ell q^{\ell+1} \\ & \equiv \left\{ u_\ell - \mathcal{D}_0(\varphi_\ell/2) - \alpha_\ell - \hat{F}_m(\mathcal{Y}_\ell) + e_0^{-1}(\sqrt{-1}\mathcal{Y}_\ell) \right\} q^{\ell+1}, \end{aligned}$$

where we put $\hat{F}(\mathcal{Y}) := \{c_1(L)^n[M]\}^{-1} 2\pi F(\sqrt{-1}\mathcal{Y})$ for each $\mathcal{Y} \in \sqrt{-1}\mathfrak{z}$. By setting $\mu_\ell := \{c_1(L)^n[M]\}^{-1} (\int_M u_\ell \omega_0^n)$, we write u_ℓ as a sum

$$u_\ell = \mu_\ell + u'_\ell + u''_\ell,$$

where $u'_\ell := (1-P)(u_\ell - \mu_\ell) \in \text{Ker } \mathcal{D}_0^\perp$ and $u''_\ell := P(u_\ell - \mu_\ell) \in \text{Ker } \mathcal{D}_0$. Now, let φ_ℓ be the unique element of $\text{Ker } \mathcal{D}_0^\perp$ such that $\mathcal{D}_0(\varphi_\ell/2) = u'_\ell$. Moreover, we put

$$\zeta_\ell := u''_\ell \quad \text{and} \quad \alpha_\ell := \mu_\ell - \hat{F}(\mathcal{Y}_\ell).$$

Then by $\mathcal{Y}_\ell = \sqrt{-1}e_0(\zeta_\ell) = \sqrt{-1}e_0(u''_\ell)$, we obtain

$$\begin{aligned} & \Phi(q; \varphi_\ell, \zeta_\ell, \alpha_\ell) \equiv \left\{ \mu_\ell + u'_\ell + u''_\ell - \mathcal{D}_0(\varphi_\ell/2) - \alpha_\ell - \hat{F}_m(\mathcal{Y}_\ell) + e_0^{-1}(\sqrt{-1}\mathcal{Y}_\ell) \right\} q^{\ell+1} \\ & \equiv \{ u''_\ell + e_0^{-1}(\sqrt{-1}\mathcal{Y}_\ell) \} q^{\ell+1} \equiv 0, \quad \text{mod } q^{\ell+2}, \end{aligned}$$

as required. Write $\sqrt{-1}\mathcal{V}/2$ as \mathcal{Y}_0 for simplicity. Now, for the real Lie subalgebra \mathfrak{b} of \mathfrak{z} generated by \mathcal{Y}_k , $k = 0, 1, 2, \dots$, its complexification $\mathfrak{b}^\mathbb{C}$ in $\mathfrak{z}^\mathbb{C}$ generates a complex Lie subgroup $B^\mathbb{C}$ of $Z^\mathbb{C}$. Then it is easy to check that the algebraic subtorus T of $Z^\mathbb{C}$ obtained as the closure of $B^\mathbb{C}$ in $Z^\mathbb{C}$ has the required properties.

Remark 6.5. In Theorem C, assume that ω_0 is a Kähler metric of constant scalar curvature, and moreover that the actions $\rho_{m(\nu)}$, $\nu = 1, 2, \dots$, coincide for all sufficiently large ν . Then by [26], the trivial group $\{1\}$ can be chosen as the algebraic subtorus T above of $Z^\mathbb{C}$.

REFERENCES

- [1] S. Bando and T. Mabuchi : *Uniqueness of Einstein Kähler metrics modulo connected group actions*, in “Algebraic Geometry, Sendai 1985” (ed. T. Oda), Adv. Stud. Pure Math. **10**, Kinokuniya and North-Holland (1987), 11–40.
- [2] N. Berline et M. Vergne : *Zeros d'un champ de vecteurs et classes caractéristiques équivariantes*, Duke Math. J. **50**(1983), 539–549.

- [3] E. Calabi : *Extremal Kähler metrics II*, in “Differential Geometry and Complex Analysis” (ed. I. Chavel, H. M. Farkas), Springer-Verlag (1985), 95–114.
- [4] D. Catlin : *The Bergman kernel and a theorem of Tian*, in “Analysis and Geometry in Several Complex Variables” (ed. G. Komatsu, M. Kuranishi), Trends in Math., Birkhäuser (1999), 1–23.
- [5] M. Cahen, S. Gutt and J. Rawnsley: *Quantization of Kähler manifolds, II*, Trans. Amer. Math. Soc. **337**(1993), 73–98.
- [6] X. Chen : *The space of Kähler metrics*, J. Differential Geom. **56** (2000), 189–234.
- [7] W.-Y. Ding: *Remarks on the existence problem of positive Kähler-Einstein metrics*, Math. Ann. **282** (1988), 463–471.
- [8] P. Deligne: *Le déterminant de la cohomologie*, Contemp. Math. J. **67** (1987), 93–177.
- [9] S. K. Donaldson: *Infinite determinants, stable bundles and curvature*, Duke Math. J. **3** (1987), 231–247.
- [10] S. K. Donaldson: *Scalar curvature and projective embeddings, I*, J. Differential Geom. **59** (2001), 479–522.
- [11] A. Fujiki : *On automorphism groups of compact Kähler manifolds*, Invent. Math. **44** (1978), 225–258.
- [12] A. Fujiki : *Moduli space of polarized algebraic manifolds and Kähler metrics*, Sugaku **42** (1990), 231–243; English translation: Sugaku Expositions **5** (1992), 173–191.
- [13] A. Futaki and T. Mabuchi: *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*, Math. Ann. **301** (1995), 199–210.
- [14] A. Futaki and T. Mabuchi : *Moment maps and symmetric multilinear forms associated with symplectic classes*, Asian J. Math. **6** (2002), 349–372.
- [15] A. Futaki and S. Morita : *Invariant polynomials of the automorphism group of a compact complex manifold*, J. Differential Geom. **21** (1985), 135–142.
- [16] D. Gieseker: *Global moduli for surfaces of general type*, Invent. Math. **43** (1977), 233–282.
- [17] S. Kobayashi : Transformation groups in differential geometry, Springer-Verlag, Berlin-Heidelberg-New York, 1972.
- [18] S. Kobayashi: *Curvature and stability of vector bundles*, Proc. Japan Acad. **58** (1982), 158–162.
- [19] A. Lichnérowicz: *Isométrie et transformations analytique d’une variété kählérienne compacte*, Bull. Soc. Math. France **87** (1959), 427–437.
- [20] Z. Lu: On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch, Amer. J. Math. **122** (2000), 235–273.
- [21] M. Lübke: *Stability of Einstein-Hermitian vector bundles*, Manuscripta Math. **42** (1983), 245–247.
- [22] H. Luo : *Geometric criterion for Gieseker-Mumford stability of polarized manifolds*, J. Differential Geom. **49** (1998), 577–599.
- [23] T. Mabuchi : *An algebraic character associated with Poisson brackets*, in “Recent Topics in Differential and Analytic Geometry,” Adv. Stud. Pure Math. **18-I**, Kinokuniya and Academic Press (1990), 339–358.
- [24] T. Mabuchi : *Vector field energies and critical metrics on Kähler manifolds*, Nagoya Math. J. **162** (2001), 41–63.
- [25] T. Mabuchi : *The Hitchin-Kobayashi correspondence for vector bundles and manifolds*, (in Japanese), Proc. 48th Geometry Symposium, Ibaraki, Aug. (2001), 461–468.
- [26] T. Mabuchi : *An obstruction to asymptotic semistability and approximate critical metrics*, to appear in Osaka J. Math. **41** (2004).
- [27] T. Mabuchi : *An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds, I & II*, preprints.

- [28] T. Mabuchi and Y. Nakagawa: *The Bando-Calabi-Futaki character as an obstruction to semistability*, Math. Annalen **324** (2002), 187–193.
- [29] T. Mabuchi and L. Weng : *Kähler-Einstein metrics and Chow-Mumford stability*, (1998), preprint.
- [30] D. Mumford, J. Fogarty and F. Kirwan: Geometric invariant theory, 3rd edition, Ergebnisse der Math. und ihrer Grenzgebiete **34**, Springer-Verlag, 1994.
- [31] D. Mumford: *Stability of projective varieties*, Enseignement Math. **23** (1977), 39–110.
- [32] Y. Nakagawa: *Bando-Calabi-Futaki characters of Kähler orbifolds*, Math. Ann. **314** (1999), 369–380.
- [33] G. Tian: *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), 99–130.
- [34] G. Tian : *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), 1–37.
- [35] K. Uhlenbeck and S.-T. Yau: *On the existence of hermitian Yang-Mills connections on stable bundles over compact Kähler manifolds*, Comm. Pure Appl. Math. **39**(1986), suppl. 257–293. and the correction **42** (1989), 703.
- [36] E. Viehweg: *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Math. und ihrer Grenzgebiete, **30**, Springer-Verlag, 1995, 1–320.
- [37] S.-T. Yau: *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), 339–411.
- [38] S. Zelditch : *Szegő kernels and a theorem of Tian*, Internat. Math. Res. Notices **6** (1998), 317–331.
- [39] S. Zhang : *Heights and reductions of semi-stable varieties*, Compositio Math. **104** (1996), 77–105.

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